AP Calculus AB   Mon, November 20, 2022

Preliminaries for FTC.

- Continuous Antiderivatives

  The book defines antiderivatives on intervals. It does not emphasize whether the intervals are closed. But for FTC, we need the endpoints.

  So, we wish antiderivatives on closed intervals to be continuous.

<u>Def</u> Let $f:[a,b]\to\mathbb{R}$. An antiderivative of f on $[a,b]$ is a function $F:[a,b]\to\mathbb{R}$

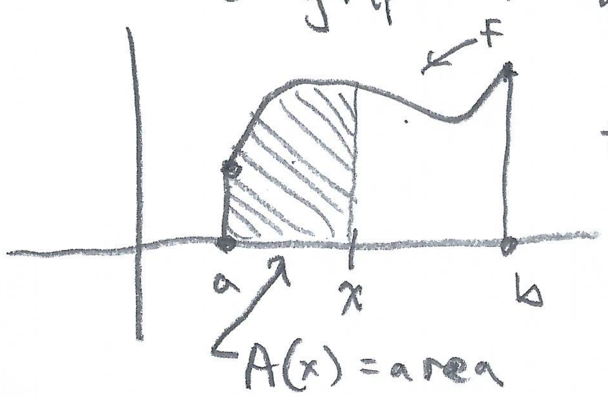
which is continuous on $[a,b]$ and differentiable on (a,b), with $F'(x)=f(x)$ for all $x\in(a,b)$.

- Dummy Variable: the variable of integration is a placeholder

  That is, $\int f(x)\,dx = \int f(t)\,dt = \int f(\text{☺})\,d\text{☺}$

- Area Function. Let $f:[a,b]\to\mathbb{R}$ be continuous.

  Define $A(x)$ to be the area bounded by the graph of f and the x-axis, on the domain $[a,x]$.



Now $A(a)=0$, but as a increases, the function A "collects" the area under the curve.

AP Calculus AB   Mon, November 20, 2022

Fundamental Theorem of Calculus

Thm (FTC I)

Let $f: [a,b] \to \mathbb{R}$ be continuous.

Define a function

$$F: [a,b] \to \mathbb{R} \quad \text{by} \quad F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$ and differentiable on (a,b), with

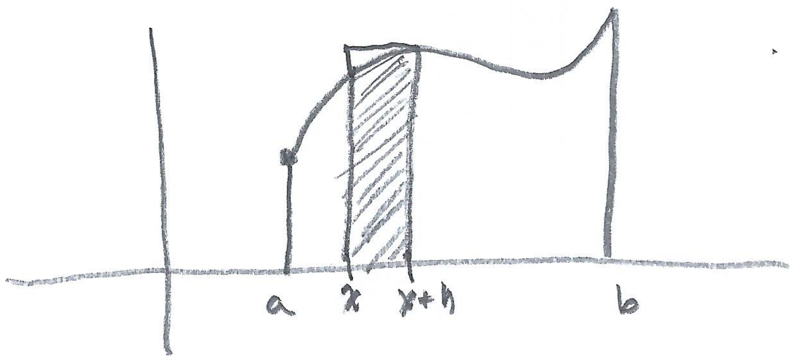
$$F'(x) = f(x) \quad \text{for all } x \in (a,b).$$

(proof) By definition,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)dt}{h}.$$

Since f is continuous, $\int_x^{x+h} f(t)dt \approx f(x)h$, and $\left| \int_x^{x+h} f(t)dt - f(x)h \right| \to 0$ as $h \to 0$. Thus $F'(x) = \lim_{h\to 0} \frac{f(x)h}{h} = f(x)$.



$\int_x^{x+h} f(t)dt \approx$ area of rectangle with height $f(x)$ and width h.

AP Calculus AB

Thm (FTC II)

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$.

Let $F:[a,b]\to\mathbb{R}$ be a continuous antiderivative for f on $[a,b]$.

Then $$\int_a^b f(x)\,dx = F(b) - F(a).$$

(proof) Define $G:[a,b]\to\mathbb{R}$ by $G(x) = \int_a^x f(t)\,dt$.

Then, by FTC I, G is a continuous antiderivative for f on $[a,b]$. Since F is also an antiderivative of f on $[a,b]$, a Corollary 2 of MVT says that

$$G(x) = F(x) + C \quad \text{for some constant } C\in\mathbb{R}.$$

Now plug in $x=a$ to get $0 = \int_a^a f(t)\,dt = G(a) = F(a) + C$, so $C = -F(a)$.

Plug in $x=b$ to get $\int_a^b f(t)\,dt = G(b) = F(b) + C = F(b) - F(a)$.

Thus, $$\int_a^b f(x)\,dt = F(b) - F(a).$$